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Coordinate transformations for magnetic fields' solid spherical harmonic coefficients

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Abstract

Magnetic particle imaging fundamentally relies on magnetic fields, so accurate characterization of the fields is crucial for various applications, such as sequence planning or model-based reconstructions. Magnetic fields can be compactly represented by solid harmonic expansions, whose coefficients efficiently describe the field distribution. However, in practice, the coordinate systems of measurement devices are not always perfectly aligned with the scanner's coordinate system, complicating direct comparison and integration of measured data from different setups. In this work, we present methods for transforming solid harmonic coefficients under translations, rotations, and point reflections. These transformations enable consistent mapping of field representations into a common coordinate system, allowing direct comparison of magnetic fields obtained in different coordinate frames.

I. Introduction

Solid harmonic expansions provide a compact representation of magnetic fields in magnetic particle imaging (MPI) for both static and dynamic cases via their expansion coefficients. Any magnetic field satisfying Laplace's equation within a ball of radius R around the origin can be expressed as

$$\boldsymbol{B}(\boldsymbol{r}) = \mathscr{S}_L(\boldsymbol{\gamma})(\boldsymbol{r}) = \sum_{l=0}^{L}\sum_{m=-l}^{l} \boldsymbol{\gamma}_{l,m} Z_l^m(\boldsymbol{r}) \quad \forall \boldsymbol{r} \in \mathscr{B}_R(\boldsymbol{0}),$$

where the $3(L+1)^2$ coefficients $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_{l,m})_{\substack{l=0,\dots,L\\ m=-l,\dots,l}} = (\gamma^x_{l,m}, \gamma^y_{l,m}, \gamma^z_{l,m})_{\substack{l=0,\dots,L\\ m=-l,\dots,l}}$ fully characterize the magnetic field $\boldsymbol{B} \in \mathbb{P}_L^3$, with $\mathbb{P}_L$ denoting the space of polynomials up to degree L [1].

However, these coefficients depend on the coordinate system of the measurement device, which my differ between scanners or field types. To transform the coefficients into a new common coordinate system, we consider three basic transformations: the translation introduced in [1], as well as the rotation and reflection through the origin presented in this work. The combination of these transformations describes any change of the coordinate system.

II. Methods and materials

Instead of transforming the magnetic field $\boldsymbol{B}$ directly, we transform its solid harmonic coefficients $\boldsymbol{\gamma}$, as illustrated by the commutative diagram

$$\begin{array}{ccc} \mathbb{R}^{3\times(L+1)^2} & \xrightarrow{\tilde{\mathfrak{F}}} & \mathbb{R}^{3\times(L+1)^2} \\ \downarrow \mathscr{S}_L & & \downarrow \mathscr{S}_L \\ \mathbb{P}_L^3 & \xrightarrow{\mathfrak{F}} & \mathbb{P}_L^3 \end{array}$$

This implies that applying the transformation $\mathfrak{F}$ to $\boldsymbol{B}$ is equivalent to applying $\tilde{\mathfrak{F}}$ to $\boldsymbol{\gamma}$, i.e., $\mathfrak{F}(\boldsymbol{B}) = \mathscr{S}_L(\tilde{\mathfrak{F}}(\boldsymbol{\gamma}))$.

Accordingly, the translation of the magnetic field by a vector $\boldsymbol{v} \in \mathbb{R}^3$ can be expressed by a translation of the coefficients, i.e., $\mathfrak{T}_{\boldsymbol{v}}(\boldsymbol{B}) = \mathscr{S}_L(\boldsymbol{\gamma}_{\text{tra}})$ with the translated coefficients

$$\boldsymbol{\gamma}_{\text{tra}} = \begin{pmatrix} \tilde{\mathfrak{T}}_{\boldsymbol{v}}(\boldsymbol{\gamma}^x) \\ \tilde{\mathfrak{T}}_{\boldsymbol{v}}(\boldsymbol{\gamma}^y) \\ \tilde{\mathfrak{T}}_{\boldsymbol{v}}(\boldsymbol{\gamma}^z) \end{pmatrix}.$$

The translation of each coefficient is given in [1].

Rotating the magnetic field with a rotation matrix $\boldsymbol{R}$ can be expressed by a rotation of the coefficients, i.e., $\mathfrak{R}_{\boldsymbol{R}}(\boldsymbol{B}) = \mathscr{S}_L(\boldsymbol{\gamma}_{\text{rot}})$ with the rotated coefficients

$$\boldsymbol{\gamma}_{\text{rot}} = \boldsymbol{R} \begin{pmatrix} \tilde{\mathfrak{R}}_{\boldsymbol{R}}(\boldsymbol{\gamma}^x) \\ \tilde{\mathfrak{R}}_{\boldsymbol{R}}(\boldsymbol{\gamma}^y) \\ \tilde{\mathfrak{R}}_{\boldsymbol{R}}(\boldsymbol{\gamma}^z) \end{pmatrix}.$$

The rotation of each coefficient is given by

$$\tilde{\mathfrak{R}}_{\boldsymbol{R}}(\boldsymbol{\gamma}) = \left(\sum_{\mu=-l}^{l} \gamma_{l,\mu} Q^l_{m,\mu}(\vartheta_1, \vartheta_2, \vartheta_3) \right)_{\substack{l=0,\dots,L \\ m=-l,\dots,l}}$$

with the rotation matrix elements $Q^l_{m,\mu}$ depending on the Euler angles $(\vartheta_1, \vartheta_2, \vartheta_3)$ (ZYZ convention) given by Collado et al. [2].

Finally, the point reflection of the magnetic field through the origin can be expressed in terms of the coefficients, i.e., $\mathfrak{P}(\boldsymbol{B}) = \mathscr{S}_L(\boldsymbol{\gamma}_{\text{pr}})$ with the reflected coefficients

$$\boldsymbol{\gamma}_{\text{pr}} = - \begin{pmatrix} \tilde{\mathfrak{P}}(\boldsymbol{\gamma}^x) \\ \tilde{\mathfrak{P}}(\boldsymbol{\gamma}^y) \\ \tilde{\mathfrak{P}}(\boldsymbol{\gamma}^z) \end{pmatrix}.$$

The point reflection of each coefficient is given by $\tilde{\mathfrak{P}}(\boldsymbol{\gamma}) = \left((-1)^l \gamma_{l,m}\right)_{\substack{l=0,\dots,L \\ m=-l,\dots,l}}$ due to the parity of the spherical harmonics [3].

III. Results and discussion

The presented coordinate transformations have been implemented in the open-source Julia package `SphericalHarmonicExpansions.jl`[1] and successfully tested on the brain scanner [4]. Figure 1 illustrates the selection fields generated by the left and right coils of the scanner. First, the coefficients are translated into the coordinate system centered at the field-free point ξ of the combined selection field. Afterward, mirroring the coefficients of the right coil's field via a combination of point reflection and rotation along the xz-plane enables a direct comparison with the left coil's field. The results demonstrate good agreement, confirming the expected symmetry between the coils.

[1] https://github.com/IBIResearch/SphericalHarmonicExpansions.jl

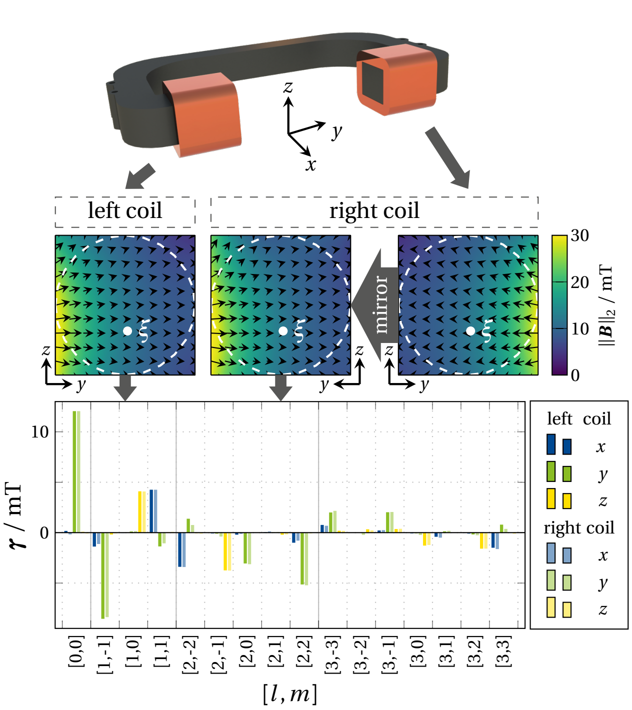

Figure 1: Selection fields of the left and right coils of the brain scanner [4]. Mirroring the right coil's field allows direct comparison with the left coil via field plots or coefficient analysis.

The proposed transformation methods consistently map solid spherical harmonic coefficients between different coordinate systems, making magnetic field representations independent of measurement setup. This allows for direct comparison of fields measured in different coordinate systems via their coefficients.

Author's statement

Conflict of interest: Authors state no conflict of interest.

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