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Randomized Greedy Kaczmarz for Multi-Contrast Magnetic Particle Imaging

Niklas Hackelberg[a,b,*] · Patricia Römer[a] · Kolya Ruppel[b] · Lina Nawwas[b,c] · Martin Möddel[b,c] · Marija Boberg[b,c] · Tobias Knopp[a,b,c]

[a] Fraunhofer Research Institution for Individualized Medical Technology and Engineering IMTE, Lübeck, Germany
[b] Institute for Biomedical Imaging, Hamburg University of Technology, Hamburg, Germany
[c] Section for Biomedical Imaging, University Medical Center Hamburg-Eppendorf, Hamburg, Germany
[*] Corresponding author, email: niklas.hackelberg@imte.fraunhofer.de

Abstract

Multi-contrast magnetic particle imaging allows for separating signals from different tracer materials or microenvironments from one joint measurement, yielding multi-channel images that distinguish tracers or encode properties such as temperature or viscosity. However, leakage artifacts enforce a noticeably longer application of iterative standard methods like the Kaczmarz algorithm to reach satisfactory reconstruction results. A greedy randomized version of the Kaczmarz method is adapted, showing the potential for faster recovery.

I. Introduction

The reconstruction task in Magnetic Particle Imaging (MPI) is to solve a linear inverse problem of the form

$$\boldsymbol{S}\boldsymbol{c} = \hat{\boldsymbol{u}}, \tag{1}$$

where $\boldsymbol{S} \in \mathbb{C}^{M\times N}$ is the system matrix, $\hat{\boldsymbol{u}} \in \mathbb{C}^M$ is the measured signal, and $\boldsymbol{c} \in \mathbb{R}^N$ represents the unknown particle distribution.

In a multi-contrast approach, $\boldsymbol{c}$ is a concatenation of vectors $\boldsymbol{c}_1 \dots, \boldsymbol{c}_\Theta$, $\Theta \in \mathbb{N}$, corresponding to different tracer concepts; see [1]. Multi-contrast MPI typically requires many iterations of methods such as the Kaczmarz algorithm to appropriately separate the different channels and achieve satisfactory reconstruction results. Motivated by this, we propose a greedy variant of the Kaczmarz method that attains a good approximation in fewer iterations. While this variant features computationally more expensive iterations, for certain parameter combinations, it can lead to runtime improvements.

II. Methods and materials

Since measurements are perturbed by noise, we consider instead of (1) a least-squares minimization problem with a Tikhonov regularization term, which is optimized by the minimum norm solution of the extended system

$$\begin{pmatrix} \boldsymbol{S} & \lambda \boldsymbol{I}_M \end{pmatrix} \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{v} \end{pmatrix} = \tilde{\boldsymbol{u}}, \tag{2}$$

with noisy measurements $\tilde{\boldsymbol{u}} \in \mathbb{C}^M$, regularization parameter $\lambda \geq 0$ and auxiliary vector $\boldsymbol{v} \in \mathbb{C}^M$. To recover the minimum norm solution of (2), we use Kaczmarz's method [2]. In the kth iteration, approximate solutions $\boldsymbol{c}^{k+1}$ and $\boldsymbol{v}^{k+1}$ are found by orthogonally projecting the respective intermediate iterate $\left(\boldsymbol{c}^k, \boldsymbol{v}^k\right)^{\mathrm{T}} \in \mathbb{C}^{N+M}$ onto the hyperplane formed by the i_kth equation of (2). The index $i_k \in \{1, \dots, M\}$ can be selected cyclically or randomly, e.g., with a probability of $\|\boldsymbol{s}_i\|_2^2/\|\boldsymbol{S}\|_F^2$ (randomized Kaczmarz method [3]), where $\boldsymbol{s}_i$ denotes the ith row of $\boldsymbol{S}$. As MPI system matrices often have few high-energy rows, the randomized variant draws indices without re-

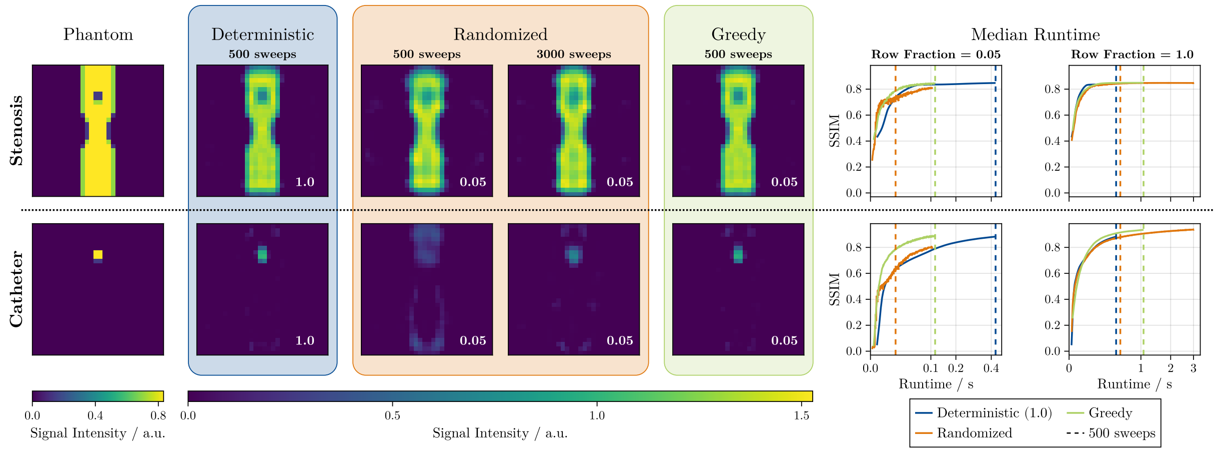

Figure 1: Reconstructions of the multi-contrast phantom described in [1] using deterministic, randomized, and greedy randomized Kaczmarz variants. Deterministic and greedy randomized runs used 500 sweeps, whereas randomized runs used 3000 sweeps. The randomized and greedy randomized variants were evaluated with row fractions of 0.05 and 1.0.

placement within one Kaczmarz sweep, where a sweep commonly means the composition of M Kaczmarz updates, i.e., a run through all equations. However, if only a subset of the rows of the system contributes significantly to the convergence of the algorithm, the number of iterations per sweep can also be a fraction of the total number of rows. In MPI, each sweep is typically followed by a projection of the current iterate $\boldsymbol{c}^k \in \mathbb{C}^N$ onto $\mathbb{R}^N_{\geq 0}$.

In this work, we extend the greedy randomized Kaczmarz variant from [4] to the common MPI setting by considering the regularized system (2), as well as the additional projections after a sweep. The greedy variant weights its index selection by the entries of the residual

$$\boldsymbol{r}^k := \tilde{\boldsymbol{u}} - \boldsymbol{S}\boldsymbol{c}^k - \lambda \boldsymbol{v}^k, \tag{3}$$

see Method 2 in [4] for more details. To account for the highest residual in every Kaczmarz update, the greedy algorithm draws indices with replacement. Note that the required update of the residual, which is obtained by

$$\boldsymbol{r}^{k+1} = \boldsymbol{r}^k - \frac{r^k_{i_k}}{\|\boldsymbol{s}_{i_k}\|_2^2 + \lambda^2}\left(\boldsymbol{S}\boldsymbol{s}^*_{i_k} + \lambda^2 \boldsymbol{e}_{i_k}\right),$$

can be accelerated by precomputing $\boldsymbol{B} := \boldsymbol{S}\boldsymbol{S}^* + \lambda^2 \boldsymbol{I}_M$. However, this update scheme does not hold after projecting onto $\mathbb{R}^N_{\geq 0}$ and, thus, after each greedy Kaczmarz sweep, the residual (3) must be fully computed.

III. Results

We compare the performance of the greedy randomized Kaczmarz algorithm with the deterministic and the standard randomized variants implemented in [5], evaluating it for a small row fraction per sweep and a full row sweep (which in the greedy case generally does not amount to a run through all rows). Using a seeded random number generator, we repeated the experiments 20 times to report median runtimes. The evaluation of the algorithms, summarized in Figure 1, reveals a good potential of the greedy method for an application with a small row fraction, resulting in faster execution times. Note that the reported runtimes do not include the precomputation times, which are approximately 4.3 seconds in our experiments, and that the observed runtime improvement can only be realized if $\boldsymbol{B}$ is precomputed.

IV. Conclusion

Our adaptation of the randomized greedy Kaczmarz method shows improvements in the tradeoff of runtime and image quality. Future work will involve investigating a greedy variant for multi-patch reconstruction. A greedy row-selection strategy could be advantageous for spatially sparse phantoms. However, precomputing $\boldsymbol{B}$ for a multi-patch system matrix $\boldsymbol{S}$ is likely to be infeasible, so a specialized operator will be needed. Furthermore, a combination of the greedy Kaczmarz method with the matrix compression approach for multi-contrast MPI presented in [6] is expected to outperform current results.

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Author's statement

Conflict of interest: Authors state no conflict of interest.

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